COLLEGIATE MATHEMATICS IN RELATION TO THE CHANGES PROPOSED IN THE SECONDARY SCHOOL COURSE.*

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Any discussion as to a reorganization of secondary-school mathematics should be of more than passing interest to teachers of collegiate mathematics. As one looks back and remembers that college teachers exercised a controlling influence on the secondary school curriculum in the respect that the various subjects—algebra, geometry, etc.—were defined by committees dominated by the college element, the interest in the present situation grows.

In the preceding paragraph I refer to the following facts.

1. Present Definitions of the Divisions of Secondary School Mathematics Laid Down Under College Influence. History of the Present Definitions.

At the summer meeting of the American Mathematical Society in 1902, a special committee was appointed by the council of the Society to report on *standard definitions of requirements for admission to colleges and scientific schools,* this committee to act in co-operation with corresponding committees of the Society for the Promotion of Engineering Education, the National Education Association, and other interested bodies. The committee appointed by the American Mathematical Society consisted of Professor H. W. Tyler (chairman), Professors Fiske, Osgood, Young, and Ziwet. The committee reported to the council of the Society on December 29, 1902, and the definitions set by it are those adopted by the College Entrance Examination Board, as specifically stated in the circular issued by the board containing the definitions of the various subjects in which exam-

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inations are set by the Board, which examinations are based upon the requirements thus defined. These definitions standardized by this committee of college teachers have stood unchanged for fourteen years. There can be no doubt from these facts that collegiate authority has dominated the content of the mathematical courses given in secondary schools. Many times I have heard earnest, progressive school men and women express discontent with this situation, and in that position I think them fully justified. If the present movement is due to some extent to a determination to reverse the order of influence, if it is a demonstration on the part of those who know secondary school boys and girls, know their natural difficulties, what they can learn and what they cannot learn, then it seems to me that a reorganization on thorough rational lines is contemplated. Without entering into well-known facts as to the astonishing change in the conditions surrounding the high-school boy and girl of to-day as compared with a decade ago, it should be self-evident to anyone who is looking for results in education that the teachers on the ground ought to and do know how mathematical instruction for which they are responsible should be conducted. Meeting many times with school men to revise examination papers set by the board, I hear constantly the statement that a certain question set by the examiners involves instruction which they have had little or no time to give. It is sufficiently obvious, at least to me, that such a question ought not be on an examination paper, and I heartily support the position of the high-school teacher in this particular. The serious earnest school man knows how much ground he can cover well, and he knows that he is sending to the higher education a better trained boy if he has been taught to do everything thoroughly, even if some of the topics in the definitions are omitted. There is no experience which reacts on a boy as favorably as becoming expert in doing the work set before him. I shall refer to this matter later.

2. Persistence of Errors in Secondary School Texts and Failure to Advance Mathematical Secondary Education by Progressive Teachers Due to Ironclad Character of Content of Courses.

In a reorganization of the secondary-school curriculum omissions should be settled by agreement of school teachers. For example, it must be clear that certain book propositions in geometry should be omitted. If teachers agree on these, their influence should prevent the appearance of these questions on examination papers. I believe that the domination of secondary mathematics by college authority has had a baleful influence upon the progress and evolution of secondary mathematics. Textbooks have become stereotyped, and the progressive author is afraid to write a different book for fear of perpetrating a freak text. The result has been that the same errors are reproduced year after year as authors follow the traditional lines. May I particularize at this point? Take the incommensurable case, that *bête-noir* of the school boy for generations. Few teachers now cover these propositions. But the reason is not, I fear, because the proofs are wrong, but because questions on this case are no longer set on college entrance papers. It is a discouraging thing to see the error inherent in the traditional proofs reproduced year after year in textbooks by authors of standing. The error has been pointed out many times, and it is an easy one to correct.

For example, in the proof of the theorem that two rectangles having equal bases are to each other as their altitudes, the mistake lies in the fact that the proof tacitly assumes that the area-number for a rectangle is proportional to the product of the dimensions, which assumption is later proved as a theorem!

Take another case, the regular polyhedrons, and look at almost any popular text. The discussion begins with the possible cases, and proves that with triangular faces, three or five can meet a vertex, etc. The proposition, however, concerns the *number of possible regular convex polyhedrons*. The argument shows for *a vertex* five possible cases. Why only five possible *polyhedrons?*

The connection is not at all obvious. A conclusion is jumped

at and the act concealed. Euler's theorem on polyhedrons, namely, that the sum of the number of vertices and the number of faces equals the number of edges increased by two, settles the difficulty at once, for this relation is linear, has one solution, and moreover, leads at once to the number of faces in any case.

For example, suppose we take the case when the faces are pentagons. Let f = number of faces. Then obviously,

$$e = \text{number of edges} = 5f \div 2,$$

$$v = \text{number of vertices} = 5f \div 3,$$

$$\therefore \frac{3}{5f} + 2 = \frac{2}{5f} + f, \quad \text{or } f = 12.$$

If the question of the number of regular polyhedrons is discussed, the usual discussion should not be called a proof.

This kind of slipshod logic is doing the school boy perhaps no harm, but the pity is to see the persistence of these errors year after year in otherwise good textbook writing. This "persistence of errors" in textbooks in geometry must be matter of comment to all progressive and alert teachers. It seems that authors fear to depart from the traditional course and exposition given in books that have sold well.

The preceding paragraphs may serve in support of my contention that the dominance of secondary mathematics by college authority has, first, *been harmful in its effect on the content and thoroughness of the teaching,* and second, *has discouraged progressive and alert teachers in experimenting on and developing new ideas with the natural outlet in publication.* We hear constantly that secondary-school teachers are, as a professional class, the most conservative people in the world. There are reasons for this other than the one of which I am speaking, some of them sufficient reasons, but what can we expect of a great body of people teaching year after year the same science, without progress, with no development possible—a dead science, in fact—and because the details of the content of this science were laid down years ago by a committe of college men?

It is because I have a deep sympathy with the progressive teacher that I take a lively interest in the discussion of changes

in the secondary-school program, and it looks to me as if school men are going to rescue secondary mathematics from the lowly condition in which college authority has placed it. If the school men can organize in this reform, can agree and say, "School mathematics is to be reformed without regard to college authority, we are going to teach algebra, geometry and the other subjects as we think these subjects should be taught, and the content of these courses we shall settle for ourselves, and they will be such as will be interesting and vital to the present-day school boy and girl"—if, I say, such a movement should develop, I believe a great service will be done for mathematical science.

What will be the attitude of college teachers in the case? What can it be but satisfied acquiescence and hearty co-operation and support? Is not the inevitable outcome of the reform that ought to be carried through going to be that a boy entering college will know what mathematics he has been taught better than he does now, because he will have been taught fewer things, and especially the hard things will have been mastered by reason of abundant time devoted to them through omission of other topics, and confidence and pride of accomplishment, the most fundamental forces in mental growth and stimulus, will have been fostered and developed? Moreover, with the content of the secondary school courses revised along the *line of human interest*—an easy matter in the case of mathematics—we shall have a different attitude and expectation on the part of college Freshmen toward mathematics. Further, the introduction of much arithmetical work in the high-school course, with insistence upon *speed and accuracy,* will have the result that boys entering college will know how to cipher, and will appreciate that an accurate location of the decimal point is a matter requiring as much care in computation problems as in fixing an employee's salary.

3. Revision of the Definitions of Entrance Requirements Necessary Before the Present Definitions Become Ridiculous.

An inevitable consequence of the present situation and discussion will be an overhauling of the definitions of elementary subjects, and this should not be entrusted to a committee com-

posed of college teachers. The growth of the number of candidates taking the Comprehensive Examination will have a bearing on this question. The line of division between elementary algebra and plane geometry is becoming less clear. Candidates are now examined on elementary mathematics, or advanced mathematics. It is not so evident, however, that any influence is at work in secondary schools tending to break down the divisions between the different topics in advanced mathematics.

If I understand the spirit underlying the present movement for reorganization, then college teachers of mathematics may expect something like the following from the Freshman of the future:

(*a*) He will ***know about*** fewer topics in the various divisions of secondary mathematics.

(*b*) He will know the fundamentals better.

(*c*) He will have confidence in approaching new mathematical subjects, and he will have an interest and eagerness in anticipating such studies.

(*d*) He will expect to be taught new subjects with thoroughness, and will believe that any topic is completed only when completely mastered. He will show by his reaction when difficulties have been overcome by patience on the part of the teacher and industry on his part, that he is conscious of a growth in mental power, and correspondingly interested and enthusiastic.

(*e*) He will expect an immense amount of strict number work in exercises in his new fields of study, for he will infer that his teachers understand that such applications are what he likes and can appreciate. He will devote himself to such applications with enthusiasm, for he will feel that his school work was directed mainly to this end, and he will feel confidence that he can "make good" in such work by virtue of the emphasis which he has himself been taught to place on speed and accuracy in computation.

(*f*) He will expect an informal introduction to new topics of difficulty, and discussion and illustration of new ideas and methods before formal proofs of theorems and formulas are insisted upon by the teacher. When he understands the content

and bearing of a new principle, he will undertake the study of the reasons for it with interest and determination.

(*g*) He will expect much repetition and drill in learning any novel technique, such as is necessary in the calculus, and he will apply himself to master this, because he will feel that it is something worth doing for himself and not because of possible scholarship penalties for failure.

The difference, then, of which the college teacher must take account, is in great part one of the attitude of mind of his students. And he must meet this and satisfy it by giving his pupils what they rightly expect. How will the teacher of collegiate mathematics meet the situation?

First, Will it be Necessary to Merge and Combine Various Divisions of Mathematics now Taught as Separate Subjects?

This question is important and is receiving much attention and thought from college men. Those who take the position that this question must be answered affirmatively advocate a year's course under the title of freshman mathematics, which should include substantially the essentials of college algebra, trigonometry, and analytic geometry. The position of these teachers is partly that of the progressive school men referred to in the above paragraphs, in that they believe that a reorganization of collegiate mathematics is desirable. Personally, I am not aware that this reorganization is actuated by any principle other than a unifying principle, which has for a basis the idea of function. In other words, the motive seems largely a need felt by college teachers for mental satisfaction in organizing various subjects into a logical consistent sequence. My belief is that Freshmen have no mental hunger of this sort to satisfy and will feel no sense of satisfaction while taking the medicine. But I do not wish it understood that I am on principle opposed to fusion. From my standpoint, fusion is of distinctly minor importance as compared with the content and underlying aim of a course, and these should agree substantially with the aims of the reorganized secondary-school course.

A condition which college teachers have to face is the decrease in the minimum requirement in mathematics for entrance

to a college course. It is a fact that 41 per cent. of our colleges and universities now require at most two units in mathematics for admission, and the tendency is to cut down this amount. It may be assumed that the low point of this tendency curve will be reached in the near future, with a following reaction to a normal, sane position. The proposed two-year course in mathematics in the junior high school seems to me a desirable minimum. Then such a reconstruction of the first year of collegiate mathematics ought to be carried out as will make the transition from school to college in the respect of mathematics smooth and continuous. Arithmetical problems will then continue to be those in which the interest can be aroused. In this connection I recall a criticism of a new text on analytic geometry reviewed a few years since by a professor in Harvard University to the effect that the "usual large number of problems of the trivial numerical sort were included by the author."

In the construction of a course in Freshman mathematics it is a matter of the greatest importance that progress in mathematical education should be a ruling principle. A question that arises in my mind in connection with the proposed first year in mathematics in the junior high school is of this nature, namely, *How much progress really has been made in mastering of methods and facts? Is the selection of material wisely done, and has the author looked forward to the future in planning the course?* There is no reason why a course complete in itself—a complete unit, in other words—cannot serve as a stepping stone to the next story in the edifice of mathematical science.

Second, With the Foundations Properly Prepared, What Changes May be Expected in the Teaching of the Calculus by Alert, Progressive Teachers?

The belief commonly accepted years ago that the calculus was inherently a difficult subject and must be reserved for the mature student has happily ceased to exist. In a recent number of the MATHEMATICS TEACHER I find a statement by a well-known educator that the first notions of differentiation and integration will be included in the elective work of the twelfth school year. A comprehensive four-year course in high-school mathematics

could, in fact, properly include simple applications of the calculus. There is in my opinion no work undertaken by the pupil in mathematics in which such a consciousness of gain in power results as from problems in maxima and minima—problems of which the real interest and importance are obvious. Many boys are willing for the first time to say that mathematics is worth something when the endless variety of problems solvable by the simple machinery of the calculus is spread out before them. They are eager to learn to run the machinery.

As I have said above, the college teacher is going to find a different attitude of mind on the part of the future student. This fact is going to influence greatly the conduct of college courses, if the teacher is to be awake to his opportunities. As I look back upon my courses in mathematics when an undergraduate and upon my experience when a young teacher, it seems to me that the spirit underlying the class work in the calculus was something like this:

"Calculus is a difficult subject and to be regarded with awe by the undergraduate. It will be impossible for many of the class to reach even a rudimentary understanding of the subject. Most of you will hardly learn to differentiate or integrate simple forms. Take the textbook and study it. The few who understand it through some unusual power of insight will be marked men—by the teacher and by the class. In other words, some of the class have natural ability in mathematics, and upon this natural ability the teacher relies. All his efforts will be directed to develop this ability in the few who are regarded as exceptions by the class as a whole."

The few students who survived such a position and were regarded as mathematicians would have done as well even under a worse system. But the student of the future who comes through the reconstructed high-school course will put an obligation on the teacher to carry him along with his co-operation to a successful finish. To accomplish this, the teacher will gradually impress him with the power of the calculus in problem work, will show the student what the calculus is for, will not worry him with little points of rigor, a matter for which the healthy boy has no taste, for he judges by results and only these. The critical faculty develops slowly in the schoolboy,

and there will be few students, perhaps none, in any class who will be prompted to doubt an obvious conclusion when connecting details are omitted. The problems taken up will be free of language which the student does not understand and in which he has not the slightest interest. In the effort to satisfy the clamor of engineering teachers for problems in the calculus which will be related to the applications, some authors of recent textbooks have made the serious mistake of crowding the pages with technical language and data which require of the teacher patient and laborious explanation, and these problems fail to arouse the slightest response because remote from the range of the student's experience.

In general terms, the course in calculus will be substantially a problem course, *learning by doing,* with reasons developed when the principle is thoroughly familiar and appreciated. To approach a new principle with ample illustration and explanation, to show the application of it with the sure result of a reaction on the student's part towards the *value* of the principle, is a certain way of arousing his curiosity as to the origin of the truth of the principle. A problem book in the calculus, with help on difficult parts, can be used as a basis for the course, supplemented by a text for purposes of reference.

In such a course reliance should be placed largely upon intuition, for the fundamental truths of the calculus need no demonstration, but merely have to be carefully explained to be believed.

Third, Will not a Supplementary Course following the Calculus Afford a Means of Unifying the Mathematical Training and Serve as Capstone of the Entire Structure?

Practising engineers rarely use the calculus, largely because it is not natural for them to do so, because they are not proficient. Algebra, geometry and trigonometry provide them with mathematical tools. These they learn to use with ease from constant practice in technical studies pursued in college courses. Assume that a general problem course could be offered after the calculus in which a large number of real questions were proposed with all convenient and suitable reference tables at hand—tables of formulas from trigonometry, calculus, numer-

ical tables, slide rules, etc. Let the emphasis be placed upon clear analysis of a problem, the comparative merit of different methods of solution, thorough discussion of results. Let the problems follow in no particular order, be entirely unclassified. Insist upon accuracy in numerical work, rejection of senseless and superfluous digits. Is it not a certainty that such a course would insure in mathematics a proficiency comparable with that resulting from laboratory work in the natural sciences?

To many it will seem that such effects upon collegiate mathematics as are described in the precedign pages may apply only to the distant future. In the article in the MATHEMATICS TEACHER referred to above is printed the following paragraph:

"Whatever may be the future of mathematics, the science will continue to be taught in the secondary schools for many years to come much as it is at present, because of the mere force of inertia if for no other reason. Schools change slowly, and the training of the necessary teachers can proceed only at a certain rate."

I am ready to accept this statement as in agreement with past experience. But I believe school men and women will move in this work of reconstruction with greater speed than tradition leads one to expect. The college teacher will gain largely by their successful efforts.

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